# Some results about Möbius functions for a finite non-solvable group

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The Möbius function associated with a finite, partially-ordered set $\chi$ (poset) $\mu_\chi$ is the map $\mu_\chi : \chi \times \chi \to \mathbb{Z}$ such that : $\mu_\chi(a,b) = 0$ unless $a \leq b$ when it is defined recursively by the equations

$$\mu_\chi(a,a) = 1$$

$$\sum_{a \leq c \leq b} \mu_\chi(a,c) = 0, \quad a < b.$$

**Generalizzation** of the classical Möbius function of the number theory:
$\mu(n) = (-1)^r$ if $n$ is the product of $r$ distinct primes
$\mu(n) = 0$ if $n$ is divisible by the square of a prime.

## Two Möbius functions related to a group $G$

If $G$ is a (finite) group, we define **two Möbius functions related to $G$, denoted by $\mu_G$ and $\lambda_G$.** Namely,

1. if $\mathcal{L}(G)$ is the subgroup lattice of $G$, the Möbius function $\mu$ of $G$ is defined as
$\mu : \mathcal{L}(G) \rightarrow \mathbb{Z}$, $H \mapsto \mu_G(H, G)(= \mu(H))$;

2. if $\mathcal{C}(G)$ is the poset of conjugacy classes of subgroups of $G$, with the order relation:
$[H] \leq [K]$ if and only if $H \leq K^g$ for some $g \leq G$,
its Möbius function is defined by
$\lambda : \mathcal{C}(G) \rightarrow \mathbb{Z}, [H] \mapsto \mu_{\mathcal{C}}([H], [G]) = \lambda(H)$

$\mu$ is the classical Möbius function, that is:

$$\mu(G) = 1, \sum_{H \leq K} \mu(K) = 0 \text{, for} H < G$$

In analogous way we define $\lambda$.

**Example** If $G$ is the cyclic infinite group, $\mu_G$ is the number theoretic Möbius function: if $G_n$ is the subgroup of $G$ of index $n$, $\mu_G(G_n) = \mu(n)$

**Does a relation between $\mu_G$ and $\lambda_G$ exist?**

**Answer: sometimes**
**We go back to 1989**: T. Hawkes, M. Isaacs, and M. Özaydin, in the beautiful paper *On the Möbius function of a finite group*, showed that

$$\mu(\{1\}) = |G'| \cdot \lambda(\{1\})$$

holds for **any finite solvable group $G$**,

More generally in **1993**, H. Pahlings ( *On the Möbius function of a finte group)*, proved that the relation

$$\mu(H) = [N_{G'}(H) : G' \cap H] \cdot \lambda(H) \tag{1}$$

is true for any $H \leq G$ whenever $G$ is **finite and solvable**.

We say that $G$ **satisfies the $(\mu, \lambda)$-property** if (1) holds for any $H \leq G$.
The $(\mu, \lambda)$-property **does not hold** for every finite group.

Not true for the Mathieu group $M_{12}$ (1990 M. Bianchi, A. Gillio and A. Verardi, *On Hawkes-Isaacs-Özaydin's conjecture*
and for the unitary groups $PSU(3, 2^{2^n})$ (2019 G. Zini, *The Möbius function of* $\mathrm{PSU}(3, 2^{2^n})$)

Here, we present some attempt to study a possible relation between $\mu$ and $\lambda$ in case of non-solvable groups

First, we suggest some reasons to justify our interest in studying $\mu$ and $\lambda$ and the relation between them.

**1. Generation of groups and probability of generating a finite group** (P. Hall(1936):

if we denote by $\sigma_n(G)$ the number of ordered $n$-tuples of elements of $G$ and by $\phi_n(H)$ the number of ordered $n$-tuples of elements of $G$ which generate $H \leq G$ respectively, we get

$$\sigma_n(G) = \sum_{H:H \leq G} \phi_n(H) \text{ and, by Möbius inversion formula}$$

$$\phi_n(G) = \sum_{H,H \leq G} \sigma_n(H)\mu(H) = \sum_{H,H \leq G} |H|^n \mu(H)$$

It follows that the probability that $n$ elements of $G$, ($G$ finite) generate $G$, $Prob_G(n)$ is given by:

$$Prob_G(n) = \frac{\phi_n(G)}{\mid G \mid^n} = \sum_{H \leq G} \frac{\mu_G(H)}{[G:H]^n}$$

**2. A complex function** (Boston, Mann 1996):

$$P_G(s) = \sum_{n \in \mathbb{N}} \frac{a_n(G)}{n^s},\ s \in \mathbb{C},$$

where

$$a_n(G) = \sum_{H \leq G, [G:H]=n} \mu_G(H)$$

$P_G(t) = Prob_G(t)$ **for** $t \in \mathbb{N}$

In case of profinite groups, the sum becomes:

$$P_G(s) = \sum_{H \leq_o G} \frac{\mu_G(H)}{| G : H |^s}$$

where $H$ ranges over all open subgroups of $G$.

Mann conjectured that this sum is **absolutely convergent** in "some" half complex plane, whenever the finitely generated profinite group is *positively generated* that is, for some $k$, the probability that $k$ random elements generate $G$, is positive. This conjecture is implied by the following 2 facts:

1. the number $|\mu_G(H)|$ is bounded by a polynomial function in $[G : H]$ ;

2. the number $b_n(G)$ of subgroups $H$ of index $n$, with $H soc(G) = G$, satisfying $\mu_G(H) \neq 0$, grows at most polynomially in $n$.

Lucchini (2010): Actually 1 and 2 are true, if they are verified for almost finite simple groups.

1 and 2 were proved in the case of alternating and symmetric groups (Lucchini-Colombo- 2010);

Not very much is known on the exact values of $\mu(H)$ when $G$ is a simple group; up to our knowledge, the only infinite families of non-abelian simple groups for which the Möbius function is completely known ( and these families verify Mann's conjecture) are the following.

- The groups $PSL(2,q)$ and $PGL(2,q)$ for any prime power $q$ (Downs 1991, Hall 1936)

- The Suzuki groups $Sz(q)$ for any odd power $q$ of 2 (Down, Jones, 2016);

- The Ree groups $Ree(q)$ for any odd power $q$ of 3 (Pierro, 2016);

- The 3-dimensional unitary groups $PSU(3,2^{2^n})$, $n > 0$ (Zini 2018);

- The groups $PSL(3,2^p)$ (Borello, D.V., Zini )

## $(\mu, \lambda)$-property for some families of non-solvable groups
(D.V., Zini)

Before considering some particular classes of groups which satisfy **$(\mu, \lambda)$-property**, we give some "general fact":

**1**: It is important the knowledge of subgroups which are intersection of maximal subgroups.
Denote by $\mathrm{MaxInt}(G)$ the set whose elements are $G$ and the subgroups of $G$ which are the intersection of maximal subgroups of $G$.
**Lemma 1.** *(Hall 1936) Let $H \leq G$ be such that $\mu(H) \neq 0$. Then $H \in \mathrm{MaxInt}(G)$.*

The same is true for $\lambda$ (note that $\mathcal{C}$ is not in general a lattice)

**Lemma 2.** *Let $H \le G$ be such that $\lambda(H) \neq 0$. Then $H \in$ MaxInt$(G)$.*

**Proof** We go by induction on $[G : H]$; The result is true for $H = G$; so, we consider $H < G$ and take $K \in$ MaxInt$(G)$ is the intersection of all maximal subgroups containing $H$. If $H$ is not in MaxInt$(G)$, $H < K$. Take $N \le G, H < N$ and $\lambda(N) \neq 0$. By induction hypothesis, $N \in$ MaxInt$(G)$ and $K \le N$. By definition of $\lambda$,

$$\lambda(H) = -\sum_{[H]<[N]\le[G],\lambda(N)\neq 0} \lambda(N) = -\sum_{[K]\le[N]\le[G]} \lambda(N) = 0.$$

we get $\lambda(H) = 0$. A contradiction.

**It follows**: If $H \le G$ and $\Phi(G) \not\le H$, then $\mu(H) = \lambda(H) = 0$

**1. Products and Extensions** - $(\mu, \lambda)$-**property** for direct products of a finite number of finite groups and for finite extensions of a finite group.

**Theorem 3.** *Let* $G = \prod_{i=1}^{n} G_i$ *be a direct product of groups* $\{G_i\}$ *such that every maximal subgroup* $M$ *of* $G$ *splits as a direct product* $M = \prod_{i=1}^{n} M_i$, *with* $M_i \leq G_i$ *for every* $i$. *If* $G_1, \ldots, G_n$ *satisfy the* $(\mu, \lambda)$*-property, then* $G$ *satisfies the* $(\mu, \lambda)$*-property.*

Theorem follows from:

**Proposition 4.** *Let* $n \geq 2$ *and* $G = \prod_{i=1}^{n} G_i$ *be a direct product of groups* $\{G_i\}$ *such that every maximal subgroup* $M$ *of* $G$ *splits as a direct product* $M = \prod_{i=1}^{n} M_i$*, with* $M_i \leq G_i$ *for every* $i$*. Let* $H = \prod_{i=1}^{n} H_i \leq G$ *with* $H_i \leq G_i$ *for every* $i$*. Then*

$$\mu_G(H) = \prod_{i=1}^{n} \mu_{G_i}(H_i), \qquad \lambda_G(H) = \prod_{i=1}^{n} \lambda_{G_i}(H_i).$$

**Proof** From the assumptions it follows immediately that, if $K \in \mathrm{MaxInt}(G)$, then $K = \prod_{i=1}^{n} K_i$ with $K_i \leq G_i$ for every $i$.

Hence, we only consider the groups $H < \prod_{i=1}^{n} K_i \leq G$ with $K_i \leq G_i$ for every $i$.

Let $I \subseteq \{1, \ldots, n\}$ be such that $H_i \neq G_i$ for $i \in I$, and $H_i = G_i$ for $i \in \{1, \ldots, n\} \setminus I$.

We have so that the subposet of $\mathcal{L}$ made by the groups $K = \prod_{i=1}^{n} K_i$ satisfying $H \leq K \leq G$ is isomorphic to the subgroup poset of groups $K = \prod_{i \in I} K_i$ satisfying $\prod_{i \in I} H_i \leq K \leq \prod_{i \in I} G_i$.

An analogous poset isomorphism holds for the posets of conjugacy classes.

Hence, $\mu_G(H) = \mu_{\prod_{i \in I} G_i}(\prod_{i \in I} H_i)$ and $\lambda_G(H) = \lambda_{\prod_{i \in I} G_i}(\prod_{i \in I} H_i)$.

Then we can assume that $H_i \neq G_i$ for all $i = 1, \ldots, n$ and we use induction on $[G : H]$.

We began considering some particular classes (**actually very small!)**

*minimal non-solvable group*: a non-solvable group whose proper subgroups are all solvable.
A minimal non-solvable group which is simple is a *minimal simple* and minimal simple groups are exactly the Frattini-free minimal non-solvable groups, as, if $G$ is a minimal non-solvable group and $\Phi(G)$ is its Frattini subgroup, then $G/\Phi(G)$ is minimal simple.

Remark: for simple groups, $(\mu, \lambda)$-**property** becomes:
$\mu(H) = [N_G(H) : H] \cdot \lambda(H)$.

## 1. minimal simple groups

- $PSL(2, 2^r)$, where $r$ is a prime;
- $PSL(2, 3^r)$, where $r$ is an odd prime;
- $PSL(2, p)$, where $p > 3$ is a prime such that $5 \mid (p^2 + 1)$;
- $Sz(2^r)$, where $r$ is an odd prime;
- $PSL(3, 3)$.

For these groups, we get the thesis by direct computation

More generally **(but not so much)**

**2. minimal non solvable groups**

Here, we consider $\Phi(G) \neq 1$ and we are left with $H \leq G$, s.t. $\Phi(G) \leq H$.
Using similar arguments to those of Pahlings, we consider $\bar{G} = G/\Phi(G)$ and $\bar{H} = H/\Phi(G)$.
We get:
$\mu(H,G) = \mu(\bar{H},\bar{G})$, $\lambda(H,G) = \lambda(\bar{H},\bar{G})$, and $[N_{\bar{G}'}(\bar{H}) : \bar{H} \cap \bar{G}'] = [N_{G'}(H) : H \cap G']$.
As $\bar{G}$ is minimal simple, the thesis follows

As I said we are considering **very small** classes. Actually, our minimal non solvable groups are particular $N$-groups (groups

whose local subgroups are all solvable). These were classified by Thompson: they are almost simple groups $G$
$S \leq G \leq \mathsf{Aut}(S)$, where $S$ is one of the following simple groups:

- the linear group $PSL(2, q)$, for some prime power $q \geq 4$;
- the Suzuki group $Sz(2^r)$, for some non-square power, $r \geq 3$
- the linear group $PSL(3, 3)$;
- the unitary group $U_3(3)$;
- the alternating group $A_7$;
- the Mathieu group $M_{11}$;
- the Tits group ${}^2F_4(2)'$.

**Example** Let $G = G_1 \times G_2$ where no non-trivial quotients of $G_1$ and $G_2$ are isomorphic. Then it is easily seen from Goursat's lemma that every maximal subgroup $M$ of $G$ splits as $M = M_1 \times M_2$ with $M_i \leq G_i$, $i = 1, 2$.

For instance, the Theorem applies to $G = G_1 \times G_2$ where $G_1$ is minimal non-solvable and $G_2$ is solvable. An easy remark about extensions:

If a group $G$ is a finite extension of a group $\bar{G}$ which does not satisfy the $(\mu, \lambda)$-property, then $G$ does not satisfy the $(\mu, \lambda)$-property.

**Remark 5.** *Let $G$ be a finite group, $H$ be a subgroup of $G$, $\mathcal{S} = \{K \leq G : H \leq K\}$ be the subposet of the subgroup lattice of $G$ made by the overgroups of $H$, and $\bar{\mathcal{S}} = \{[K] \leq [G] : [H] \leq$*

$[K]\}$ *be the corresponding subposet of the conjugacy classes* $[K]$ *with* $[H] \leq [K]$*. Suppose that, for every* $K \in \mathcal{S} \setminus \{G\}$*, we have* $N_{G'}(K) = G' \cap K$*. Then the* $(\mu, \lambda)$*-property for* $H$ *holds if and only* $\mu(H) = \lambda(H)$*, and hence, if and only if the posets* $\mathcal{S}$ *and* $\bar{\mathcal{S}}$ *are isomorphic. In the case* $S_4 \cong H \leq G = U_3(3)$*, we have that* $\mathcal{S} = \{H, M_1, M_2, M_3, G\} \ncong \bar{\mathcal{S}} = \{[H], [M_1], [M_2] = [M_3], [G]\}$*, where* $M_i$ *is a maximal subgroup of* $G$*. For every* $K \in \mathcal{S}$*,* $K$ *is self-normalizing in the simple group* $G$*. Thus,* $\mathcal{S} \ncong \bar{\mathcal{S}}$ *implies that the* $(\mu, \lambda)$*-property fails at* $H$*.*

**Thank You**

## Some Bibliography

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